PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. Find all the pure and mixed strategy Nash Equilibria (NE) of the following game.

		Player 2		
		L	C	R
Player 1	U	$0, 1$	$1, 10$	$1, 2$
	D	$1, 0$	$0, 0$	$1, 1$

Note: To fix notation, let p be the probability with which player 1 plays U, let r be the probability with which player 2 plays L, and let q be the probability with which player 2 plays C.

Solution: R strictly dominates L, so we can eliminate it. The new game looks as follows:

		Player 2	
		C	R
Player 1	U	$\underline{1}, \underline{10}$	$\underline{1}, 2$
	D	$0, 0$	$\underline{1}, \underline{1}$

Recall that p is the probability with which player 1 plays U, and q is the probability with which player 2 plays C.

Calculate the best responses. For player 1 (the best response indicates the optimal value of p):

$$BR_1(q) = \begin{cases} 1 \text{ if } q > 0, \\ [0, 1] \text{ if } q = 0. \end{cases}$$

Calculate the best responses. For player 2 (the best response indicates the optimal value of q):

$$BR_2(p) = \begin{cases} 1 \text{ if } p > 1/9, \\ [0, 1] \text{ if } p = 1/9 \\ 0 \text{ if } p < 1/9. \end{cases}$$

The two pure-strategy equilibria give us $(p, q, r) = (1, 1, 0)$ and $(0, 0, 0)$. This corresponds to (U, C) and (D, R), respectively. The mixed-strategy equilibria are $(p, q, r) = (p, 0, 0)$ for $p \leq 1/9$.

2. Two tech entrepreneurs have made 1 dollar through a new app and need to decide how to allocate the gains. If they can't agree, nobody gets anything. Let x_1 and x_2 be the amounts that entrepreneur 1 and 2 get. Their payoffs are:

$$\begin{aligned} u_1(x_1) &= x_1^2 \\ u_2(x_2) &= x_2. \end{aligned}$$

(a) Calculate U, the set of possible payoff pairs. Can the symmetry axiom (SYM) be used to conclude that the Nash Bargaining Solution must satisfy $v_1^* = v_2^*$? Why/why not? (1 sentence).

> **Solution:** From the problem description $x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 1$, and disagreement allocation $D = (0,0)$. Using the inversions $x_1 = \sqrt{v_1}$ and $x_2 = v_2$ we get $U = \{(v_1, v_2) | v_1, v_2 \geq 0, \sqrt{v_1} + v_2 \leq 1\}$. The disagreement payoff is $d = (0^2, 0) = (0,0)$. Since the players are not symmetric, we cannot apply the symmetry axiom.

(b) Find the Nash Bargaining Solution. What are the allocations? That is, how much money does each entrepreneur get?

> **Solution:** We can solve the program $\max_{v_1,v_2}(v_1 - d_1)(v_2 - d_2)$ subject to $(v_1, v_2) \in U$. The solution must be efficient, so we can substitute $\sqrt{v_1} + v_2 = 1$ into the problem, along with $d_1 = d_2 = 0$. Thus: $(v_1 - d_1)(v_2 - d_2) = v_1 v_2 = v_1(1 - \sqrt{v_1})$. Take the first-order condition: $1 - \frac{3}{2}\sqrt{v_1} = 0$. This gives $v_1^* = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Then $v_2^* = 1 - \sqrt{v_1^*} = \frac{1}{3}$. As can be checked, this corresponds to the allocations $x_1^* = \frac{2}{3}$ and $x_2^* = \frac{1}{3}$.

(c) Suppose now that the entrepreneurs have signed a contract such that in case of disagreement, entrepreneur 2 gets to keep 0.5 dollar whereas entrepreneur 1 gets nothing. What is the new disagreement point? Find the Nash Bargaining Solution. What are the allocations?

> **Solution:** The new disagreement payoffs are $d = (0^2, 1/2) = (0, 1/2)$. The Nash product is now $(v_1 - 0)(v_2 - 1/2) = v_1(1 - \sqrt{v_1} - 1/2) = v_1(1/2 - \sqrt{v_1})$. First-order condition: $1/2 - \frac{3}{2}\sqrt{v_1} = 0$. Thus $v_1^* = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$. Then $v_2^* = 1 - \sqrt{v_1^*} = \frac{2}{3}$. This corresponds to the allocations $x_1^* = \frac{1}{3}$ and $x_2^* = \frac{2}{3}$.

(d) Compare the allocations in (c) to those in (b), and comment on any difference you find.

> **Solution:** Now, entrepreneur 2 gets the highest allocation. The reason is that he has a better bargaining position in (c) than in (b) due to the disagreement point: now, entrepreneur 2 is assured at least an allocation of 1/2, and he can use that to bargain for a further 2/3-1/2=1/6 of the total gains.

3. Suppose we are in a **private value** auction setting. There are two bidders, $i = 1, 2$. They have valuation v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim U(1, 2).$$

The auction format is **sealed-bid first price**. In case of a tie, a fair coin is flipped to determine the winner.

(a) Suppose player j uses the strategy $b(v_j) = cv_j + d$, where c and d are constants. Show that if bidder $i \neq j$ bids b_i, his probability of winning is

$$\mathbb{P}(i \text{ wins}|b_i) = \frac{b_i - d - c}{c},$$

whenever $c + d \leq b_i \leq 2c + d$.

Hint: Recall that if $x \sim U(a, b)$ then $\mathbb{P}(x \leq y) = \frac{y-a}{b-a}$ for $y \in [a, b]$.

Solution: Notice that for $c + d \leq b_i \leq 2c + d$:

$$\mathbb{P}(i \text{ wins}|b_i) = \mathbb{P}(b_i \geq b(v_j)) = \mathbb{P}(v_j \leq (b_i - d)/c) = \left(\frac{b_i - d}{c} - 1\right) = \frac{b_i - d - c}{c}.$$

(b) Using the result in (a), show that there is a symmetric Bayesian Nash equilibrium (BNE) in linear strategies $b(v_i) = cv_i + d$, $i = 1, 2$. Find c and d.

Solution: The expected payoff of bidder i if he bids b_i and bidder j bids according to the equilibrium strategy is

$$\mathbb{P}(i \text{ wins}|b_i)\,[v_i - b_i] = \left(\frac{b_i - d - c}{c}\right)[v_i - b_i]\,.$$

The first-order condition with respect to b_i is

$$\frac{1}{c}\,[v_i - b_i - (b_i - d - c)] = 0.$$

This yields $b_i = \frac{1}{2}[v_i + c + d]$. Matching coefficients we get $c^* = 1/2$ and $d^* = \frac{1}{2}(c^* + d^*) = \frac{1}{2}(\frac{1}{2} + d^*)$ which implies $d^* = \frac{1}{2}$.

(c) Now suppose instead that we are in a **common value** auction setting. The auction format is still **sealed-bid first price**. Thus, the object has common value $v_1 = v_2 = v$. We assume that

$$v = 1 + s_1 + s_2,$$

where s_1 and s_2 are independently distributed according to

$$s_i \sim U(0, 1/2).$$

Bidder i observes only s_i, but not s_j. Show that there is a symmetric BNE in which both bidders use the strategy $b(s_i) = cs_i + d$, $i = 1, 2$, and find c and d.

Solution: Notice that for $d \le b_i \le \frac{c}{2} + d$:

$$\begin{aligned}\mathbb{P}(i \text{ wins}|b_i) &= \mathbb{P}(b_i \ge b(s_j)) \\ &= \mathbb{P}(s_j \le (b_i - d)/c) \\ &= 2 \cdot \frac{b_i - d}{c}.\end{aligned}$$

Similarly, we can find the expected value of s_j conditional on winning.

$$\begin{aligned}\mathbb{E}[s_j|i \text{ wins}, b_i, s_i] &= \mathbb{E}[s_j|b_i \ge b(s_j)] \\ &= \mathbb{E}[s_j|b_i \ge cs_j + d] \\ &= \mathbb{E}\left[s_j|s_j \le (b_i - d)/c\right] \\ &= \frac{b_i - d}{2c}.\end{aligned}$$

Taking bidder j's strategy $b(s_j) = cs_j + d$ as given, the expected utility to bidder i from bidding b_i is then

$$\begin{aligned}\mathbb{E}[u_i(b_i, b_j^*)] &= \mathbb{P}(i \text{ wins}|b_i)\left(\mathbb{E}[v|i \text{ wins}, b_i, s_i] - b_i\right) \\ &= 2 \cdot \frac{b_i - d}{c}\left(1 + s_i + \mathbb{E}[s_j|i \text{ wins}, b_i, s_i] - b_i\right) \\ &= 2 \cdot \frac{b_i - d}{c}\left(1 + s_i + \frac{b_i - d}{2c} - b_i\right)\end{aligned}$$

Take the FOC with respect to b_i:

$$\frac{2}{c}\left(1 + s_i + \frac{b_i - d}{2c} - b_i\right) + 2 \cdot \frac{b_i - d}{c}\left(\frac{1}{2c} - 1\right) = 0.$$

This yields $b_i = \frac{c}{2c-1}s_i + \frac{cd+c-d}{2c-1}$. Thus, $c^* = c^*/(2c^* - 1)$ which yields $c^* = 1$. Similarly, $d^* = \frac{c^*d^*+c^*-d^*}{2c^*-1} = \frac{1 \cdot d^*+1-d^*}{2 \cdot 1-1}$ which yields $d^* = 1$.